Scalar-Fluid theories: cosmological perturbations and large-scale structure

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ABSTRACT: Recently a new Lagrangian framework was introduced to describe interactions between scalar fields and relativistic perfect fluids. This allows two consistent generalizations of coupled quintessence models: non-vanishing pressures and a new type of derivative interaction. Here the implications of these to the formation of cosmological large-scale structure are uncovered at the linear order. The full perturbation equations in the two cases are derived in a unified formalism and their Newtonian, quasi-static limit is studied analytically. Requiring the absence of an effective sound speed for the coupled dark matter fluid restricts the Lagrangian to be a linear function of the matter number density. This still leaves new potentially viable classes of both algebraically and derivatively interacting models wherein the coupling may impact the background expansion dynamics and imprint signatures into the large-scale structure.

KEYWORDS: Scalar-fluid theory, interacting dark energy, perturbations, large-scale structure

Contents

1 Introduction

It is generally believed that in order to describe the complete universe history, from its very early stages, namely the inflationary epoch or alternatively the bouncing regime, up to the late-time accelerated era, one needs to introduce additional degrees of freedom, beyond either general relativity or the standard model of particle physics. If these new fields are of gravitational origin then we may refer to "modified gravity" [1, 2], which exhibits general relativity as a particular limit; and if the extra degrees of freedom belong to the matter content of the universe, we have "dark energy" and "dark matter" [3, 4], although one can construct combined scenarios of the above extensions, for instance introducing various

non-minimal couplings, and typically shift between them, partially or completely, using suitable transformations.

There are numerous proposed realizations of the dark sector, but most of them allow at least an effective description in terms of scalar fields. Dark matter is usually modeled as a cold (i.e. pressureless) perfect fluid component, and in its simplest and prototypical form, dynamical dark energy can be modelled as a "quintessence" [5–7] scalar field rolling down its potential, as the inflaton field in the early universe at vastly higher energy scales. Generalizations include non-canonical kinetic terms [8–11] and non-minimal couplings to e.g. the Ricci curvature [12–15], the Gauss-Bonnet invariant [16–18], torsion and nonmetricity [19–21], or the derivatives of the field [22–24]. The most general viable action would encompass the Horndeski-type [25, 26] and even more general [27, 28] theories. From a particle physics point of view, we generically expect the new scalar degree(s) of freedom to interact with each other and with (dark) matter [29–42]. Such an interaction could in fact potentially alleviate the coincidence problem, namely the comparability of the dark matter and dark energy densities today may be more naturally explained if they don't scale independently throughout the evolution of the universe.

As the microscopic theory of the dark sector is completely undisclosed, it is useful to seek a general parameterization of the possible cosmological implications of theoretically consistent models. Recently, a new approach to constructing interacting theories of the dark sector was presented in [43, 44]. There, dark matter is described as a perfect fluid, using Brown's Lagrangian formulation of relativistic fluids that employs a set of Lagrange multipliers [45]; see Refs. [46, 47] for applications of the same formalism to non-minimally coupled theories. By adding a scalar field into the theory and allowing suitable non-minimal couplings with the fluid variables, in particular either the fluid's density functional or its four-velocity, one then obtains classes of theories in which the interacting dark matter component has unusual properties compared to the previously studied coupled quintessence models. We call these new theories *the Scalar-Fluid theories*. An immediate benefit of the Lagrangian formalism is that the system of conservation equations is automatically satisfied, and given the form of the Lagrangian, the interaction terms are uniquely determined up to all orders. This is not the case in the approach adopted in e.g. Refs. [48–57], where the phenomenological parameterisation of the coupling terms directly in the conservation equations has to be done "by hand" at both the levels of background and of perturbations[1]. Nevertheless systematical, general parameterisations of coupled dark sector cosmologies have been very recently constructed within the frameworks of effective field theory [60], pull-back-formalism for fluids [61], and the parameterised post-Friedmannian formalism [62].

In this paper we investigate the behavior of the scalar-fluid cosmological scenarios at the perturbation level. In particular, we uncover the implications of the new couplings

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[1]Naive phenomenological parameterisations may fail to capture the physics of more fundamental approaches (as one may argue on general grounds [58] and see directly by comparing with the results derived from explicit high energy physics theories [59]); furthermore, unphysical instabilities [51, 52] or even inconsistencies with the covariant stress energy conservation [32, 48] are possible unwanted artefacts of such parameterisations.

these models feature, to the large scale structure formation, with the aim of confronting the theories with the high-precision data on the galaxy distributions and gravitating matter sources in the universe. To begin, we will review the theoretical framework of Refs. [43, 44] in section 2. We shall then first present the cosmological equations in Section 3 and in Section 4 derive their relevant quasi-static limit for the models with algebraic coupling between the dark components, and separately, analyze the qualitatively different scenarios with derivative couplings. In both cases we are able to understand the generic features of linear perturbation evolution in these classes of models already by our analytic considerations. Conclusions and perspectives that emerge from our analysis are provided in section 5.

Some additional material is confined to the three appendices: in A we present the perturbation equations in an alternative formulation, in B we sketch a convenient approach to solve the perturbations equations numerically making some first steps towards full numerical analysis of the new models, and finally in C we consider, from the point of view of high-energy physics motivated couplings, some natural generalisations of the scalar-fluid theories.

2 Scalar-fluid theories

In this section we briefly review scalar-fluid theories, in which the fluid and the scalar field describe the dark matter and dark energy sectors respectively, based on [43, 44]. In the first subsection we briefly review the basic formalism, then in 2.1.2 we generalize it to the cases of algebraic and derivative couplings, and finally we extract the gravitational field equations in 2.2.

2.1 Lagrangian formulation of a scalar field interacting with a relativistic fluid

We will employ the Lagrangian formalism for relativistic fluids as presented in Ref. [45], since it is complete in the sense of incorporating thermodynamics too. Alternative Lagrangian formulations have been developed by many authors, see for example Ref. [63] for an earlier and somewhat simpler formulation and Refs. [64–66] for a more general framework of an effective theory of fluids.

2.1.1 Minimal theories

In order to construct a Lagrangian of a scalar field interacting with a relativistic fluid, let us first start from the Lagrangian of a minimally coupled relativistic fluid following [45]. Such a Lagrangian can be expressed as

$$\mathcal{L}_M^{(minimal)} = -\sqrt{-g}\,\rho(n,s) + J^\mu \left(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha_{,\mu}^A \right) , \qquad (2.1)$$

where $g_{\mu\nu}$ is the metric, with g its determinant. In the above expression $\rho(n,s)$ is the energy density of the matter fluid, assumed to be a function of the particle number density n and the entropy density per particle s. Moreover, we need to introduce the Lagrange

multipliers φ, θ and β_A, with $A = 1, 2, 3$, and where α_A are the Lagrangian coordinates of the fluid. The vector-density particle-number flux J^μ obeys the following relations:

$$J^\mu = \sqrt{-g}\, n\, U^\mu , \qquad |J| = \sqrt{-g_{\mu\nu} J^\mu J^\nu} , \qquad n = \frac{|J|}{\sqrt{-g}} , \tag{2.2}$$

with U^μ the fluid 4-velocity satisfying $U_\mu U^\mu = -1$. In summary, the independent dynamical variables of the Lagrangian (2.1) are $g^{\mu\nu}$, J^μ, s, φ, θ, β_A and α^A, and the corresponding variations will give rise to the field equations.

Variation with respect to the metric $g^{\mu\nu}$ gives the energy-momentum tensor

$$T_{\mu\nu} = \rho\, U_\mu U_\nu + \left(n \frac{\partial\rho}{\partial n} - \rho \right) (g_{\mu\nu} + U_\mu U_\nu) , \tag{2.3}$$

which can be rewritten in the usual perfect-fluid form

$$T_{\mu\nu} =\equiv \frac{-2}{\sqrt{-g}} \frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} = p\, g_{\mu\nu} + (\rho + p)\, U_\mu U_\nu , \tag{2.4}$$

if we identify the pressure p as

$$p = n \frac{\partial\rho}{\partial n} - \rho . \tag{2.5}$$

Additionally, variations with respect to the remaining dynamical variables lead to:

$$J^\mu : \qquad \mu\, U^\mu + \varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu} = 0 , \tag{2.6}$$

$$s : \qquad -\frac{\partial\rho}{\partial s} + n\, U^\mu \theta_{,\mu} = 0 , \tag{2.7}$$

$$\varphi : \qquad J^\mu_{,\mu} = 0 , \tag{2.8}$$

$$\theta : \qquad (s J^\mu)_{,\mu} = 0 , \tag{2.9}$$

$$\beta_A : \qquad J^\mu \alpha^A_{,\mu} = 0 , \tag{2.10}$$

$$\alpha^A : \qquad (J^\mu \beta_A)_{,\mu} = 0 , \tag{2.11}$$

with

$$\mu = \frac{\rho + p}{n} = \frac{\partial\rho}{\partial n} , \tag{2.12}$$

the chemical potential, and where a comma denotes a partial derivatives ∂_μ. Relations (2.8) and (2.9) respectively correspond to the particle number conservation and the entropy exchange constraint, which is consistent with the fact that φ and θ are Lagrange multipliers. They can be re-expressed as

$$\nabla_\mu (n\, U^\mu) = 0 , \tag{2.13}$$

$$\nabla_\mu (s\, n\, U^\mu) = 0 , \tag{2.14}$$

where ∇_μ denotes the covariant derivative. Similarly, according to (2.10), the three Lagrange multipliers β_A constrain the fluid's 4-velocity to be oriented along the flow lines with constant α^A. Note that combining (2.7), (2.10) and (2.11) allows us to extract the chemical free energy as

$$F = \mu - T\, s = U^\mu \varphi_{,\mu} . \tag{2.15}$$

Using the above field equations one can easily show that the fluid energy-momentum tensor is conserved, namely $\nabla_\mu T^{\mu\nu} = 0$.

At this point, let us for concreteness consider some examples of fluids. The most important characteristic is the equation of state w, defined by $p = w\rho$. A barotropic fluid with the constant equation of state w_0 is modelled simply as $\rho(n) \sim n^{1+w_0}$. A more nontrivial equation of state is that of a "Chaplygin gas", $p = -A/\rho^{-\alpha}$, with some positive constants A and α [67, 68]: this corresponds to $\rho = \left(A + Bn^{1+\alpha}\right)^{\frac{1}{1+\alpha}}$, for a suitable B. The so called Cardassian models lead to similar expansion histories without separating dark matter and dark energy [69], and the corresponding fluid description would be given by $\rho \sim n\left(1 + An^{-q\nu}\right)^{\frac{1}{q}}$, where A is again a constant of appropriate dimensions and q and ν are dimensionless parameters.

Finally, imposing equations (2.6)-(2.11) into (2.1) one can obtain the on-shell Lagrangian of the fluid as [43]

$$\mathcal{L}_M = -\sqrt{-g}\,\rho = \sqrt{-g}\,p \quad \text{(on-shell)}, \tag{2.16}$$

where the last equality holds up to total derivatives and only once the equations of motion of the non-interacting fluid have been taken into account. Thus, this will not hold in general in the following, where we will consider non-minimal couplings of the fluid variables[2].

2.1.2 Non-minimal theories

Having obtained the Lagrangian for the dark matter fluid, we can now proceed to the construction of the total action of scalar-fluid theories, which includes an interaction between the matter fluid and the scalar field that describe the dark energy sector. In particular, the total action consists of three pieces:

$$\mathcal{S} = \int d^4x \left[\mathcal{L}_{\text{grav}} + \mathcal{L}_M + \mathcal{L}_\phi\right]. \tag{2.17}$$

In the above expression $\mathcal{L}_M$ is the perfect-fluid Lagrangian (2.1) that is now generalized to include interactions with the scalar field, $\mathcal{L}_{\text{grav}}$ is the standard Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\text{grav}} = \frac{\sqrt{-g}}{2\kappa^2}R, \tag{2.18}$$

with R the Ricci scalar and κ the gravitational coupling, related to the Newton's constant G through $\kappa^2 = 8\pi G$, and where $\mathcal{L}_\phi$ is the free scalar-field Lagrangian, which we take here for simplicity to correspond to a canonical scalar field ϕ with a potential $V(\phi)$, namely

$$\mathcal{L}_\phi = -\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right]. \tag{2.19}$$

In the total action (2.17) we have included the interactions in the matter Lagrangian $\mathcal{L}_M$, which thus incorporates the coupling of the fluid and scalar degrees of freedom. The energy

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[2]In the context of non-minimal curvature-matter couplings [70–72], the above observation clarifies the pseudo-issue of the choice of the matter Lagrangian; see discussions and references in section 2.2 of the review [72].

density ρ of matter for example corresponds then to the total energy density including the contribution from the interaction with the field. In principle the separation into the fluid and field terms is arbitrary in the coupled case [58], but lumping all the matter-coupled terms besides the canonical scalar terms in (2.19) to the "fluid sector" appears both physically intuitive and technically more convenient[3]. Moreover, for clarity, we will focus separately on the two distinct generalizations (though in principle both could be present simultaneously):

- Algebraic coupling [43]:
$$\mathcal{L}_M = -\sqrt{-g}\,\rho(n,s,\phi) + J\,. \tag{2.20}$$

- Derivative coupling [44]:

$$\mathcal{L}_M = -\sqrt{-g}\,\rho(n,s) + f(n,s,\phi)J^\mu\partial_\mu\phi + J\,. \tag{2.21}$$

Here J is a shorthand notation for the second term in the RHS of (2.1). In both (2.20) and (2.21), $f(n,s,\phi)$ is an arbitrary function of the scalar field ϕ, of the fluid particle number density n and of the entropy density per fluid particle s, and thus one has an implicit coupling between the fluid dynamical degrees of freedom (i.e excluding the various Lagrange multipliers) and the scalar field. The difference between (2.20) and (2.21) is that the former represents a general coupling in the absence of derivatives, while the latter is a coupling that one can construct in the presence of scalar-field derivatives, which should then need to be coupled with the the fluid vector-density particle number flux J^μ.

In summary, the total action (2.17) describes theories where the new scalar-fluid interactions stem from the new field-dependent terms (2.20) and (2.21) implemented in the Lagrangian variational approach to relativistic fluids. The functions ρ and f in the action determine the coupling completely, the system is automatically consistent with the Bianchi identities, and there are no ambiguities to be fixed by hand, unlike in models based on phenomenological parameterizations of the conservation equations.

2.2 Field equations

Let us now use the total action (2.17) of scalar-fluid theories, in order to extract the field equations, by varying with respect to $g^{\mu\nu}$, ϕ, J^μ, s, φ, θ, β_A and α^A. We will do this in a unified notation and specify the relevant terms in the equations of motion separately for algebraic couplings [43], i.e for the interacting Lagrangians (2.20), and for derivative couplings [44], i.e for the interacting Lagrangians (2.21).

In the analysis below, we will neglect the role of the entropy density s. In fact from equations (2.13) and (2.14), one finds that in general

$$\partial_\mu s = 0\,, \tag{2.22}$$

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[3]We present the equations in an alternative formalism that isolates the coupling terms as $\mathcal{L}_M = \mathcal{L}_M^{(minimal)} + \mathcal{L}_{\text{int}}$, in appendix A. This coincides with the notation originally used in [43, 44]. In the following we will however use the simpler notation of Eq. (2.17).

a relation that holds regardless of the couplings. This implies that the entropy density is always a constant and we have an adiabatic process. Did one want to include entropic effects, one needed to modify the constraints by adding dynamics to s in the Lagrangian. However, as we focus here on the effects of the couplings we will for simplicity neglect the possible intrinsic entropy of the fluid. Hence, the functions coupling the scalar to the fluid in (2.20) and (2.21) can be considered to depend only on n and ϕ in what follows.

The field equations given by the variation with respect to the metric, retain their usual form, namely

$$G_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} + T^{(\phi)}_{\mu\nu} \right) , \qquad (2.23)$$

where we recall from (2.4) that

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} , \qquad (2.24)$$

and similarly for the scalar field, for which the canonical Lagrangian (2.19) gives

$$T^{(\phi)}_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{L}_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + V(\phi) \right] . \qquad (2.25)$$

The crucial issue is definitely that the components of the energy momentum tensor $T_{\mu\nu}$ (where for simplicity we have omitted the superscript M for the matter fluid), depend also upon the scalar field (and possibly its first derivatives). Consequently, the individual energy momentum tensors are not conserved, though their sum, due to the diffeomorphism invariance of the matter action and in accordance with the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, is, namely

$$\nabla^\mu \left(T_{\mu\nu} + T^{(\phi)}_{\mu\nu} \right) = 0 . \qquad (2.26)$$

Following the usual notation, we can parametrize the interaction with a four-vector Q_μ as

$$\nabla^\mu T^{(\phi)}_{\mu\nu} = Q_\nu = -\nabla^\mu T_{\mu\nu} . \qquad (2.27)$$

It remains to deduce the non-conservation terms Q_μ in the two cases we are considering.

We obtain, respectively, the following results:

- Algebraic coupling (Eq. (2.20)):

$$
\begin{aligned}
\rho &= \rho(n, \phi) , \\
p &= n \frac{\partial \rho(n, \phi)}{\partial n} - \rho(n, \phi) , \\
Q_\nu &= \frac{\partial \rho(n, \phi)}{\partial \phi} \partial_\nu \phi .
\end{aligned}
\qquad (2.28)
$$

- Derivative coupling (Eq. (2.21)):

$$
\begin{aligned}
\rho &= \rho(n) , \\
p &= n \frac{\partial \rho(n)}{\partial n} - \rho(n) - n^2 \frac{\partial f(n, \phi)}{\partial n} U^\lambda \partial_\lambda \phi , \\
Q_\nu &= -n^2 \frac{\partial f(n, \phi)}{\partial n} \nabla_\lambda U^\lambda \partial_\nu \phi .
\end{aligned}
\qquad (2.29)
$$

In both cases, the Klein-Gordon equation follows from (2.27) and (2.25) as

$$\left(\Box\phi - V'\right)\partial_\nu\phi = Q_\nu\,. \tag{2.30}$$

The stress energy conservation equations follow from (2.27) and (2.4), and can be written as

$$U^\mu\nabla_\mu\rho + (\rho + p)\nabla_\mu U^\mu = U^\mu Q_\mu\,, \tag{2.31}$$

$$(\rho + p)\,U^\nu\nabla_\nu U^\mu + U^\mu U^\nu\nabla_\nu p + \nabla^\mu p = -\left(g^{\mu\nu} + U^\mu U^\nu\right)Q_\nu\,, \tag{2.32}$$

when projected along the fluid flow and orthogonal to it, respectively. Our notation slightly differs from Refs. [43, 44], however the results coincide, and we refer the reader to those works for more details on the derivations.

3 Cosmological equations

In the previous section we presented the formulation of scalar-fluid theories, in which one incorporates the scalar-fluid interaction, i.e. the interaction between dark energy and dark matter, straightaway at the Lagrangian level, instead of inserting it by hand at the level of equations of motion. We extracted the covariant field equations, which can be used in the relevant applications. Here we specialize to cosmology, first deriving the equations of motion adapted to the background describing the overall cosmic expansion, and then to linear perturbations, describing the formation of cosmological large-scale structure.

3.1 Background

In particular, in order to explore the cosmological applications of the above new class of theories, we focus on the Friedmann-Robertson-Walker (FRW) metric of the form

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right)\,, \tag{3.1}$$

with $a(t)$ the scale factor, $K = -1, 0, 1$ for spatially open, flat or closed geometry respectively, and with $d\Omega^2$ the two-dimensional line element of a sphere. Additionally, all the variables that appear in the equations of the previous sections are assumed to be homogeneous, that is depending only on the cosmic time t. Finally, concerning the fluid 4-velocity, in comoving coordinates it becomes $U^\mu = (1, 0, 0, 0)$.

In the FRW geometry, the Friedmann equations of the scalar-fluid theories read [43, 44]

$$3\frac{K}{a^2} + 3H^2 = \kappa^2\left(\rho + \frac{1}{2}\dot\phi^2 + V\right)\,, \tag{3.2}$$

$$\frac{K}{a^2} + 2\dot H + 3H^2 = -\kappa^2\left(p + \frac{1}{2}\dot\phi^2 - V\right)\,, \tag{3.3}$$

where the dot denotes derivative with respect to t and we have introduced the Hubble parameter $H = \dot a/a$. As expected, these equations retain their usual form. Moreover, we obtain the conservation equation for the number density as

$$\dot n + 3Hn = 0 \quad \Rightarrow \quad n \sim a^{-3}\,. \tag{3.4}$$

Had we kept the entropy s in the system, we would have recovered its conservation equation at the FRW background as just $\dot{s} = 0$. Finally, the conservation equations feature the couplings as follows:

$$\dot{\rho} + 3H(\rho + p) = Q_0 \,, \tag{3.5}$$

$$\ddot{\phi} + 3H\dot{\phi} + V' = -Q_0/\dot{\phi} \,. \tag{3.6}$$

In these conservation equations the scalar-fluid interaction is taken into account by the time-component of the coupling vector Q_μ, namely Q_0. In the two cases we consider, the coupling turns out as:

- Algebraic coupling (Eqs. (2.20),(2.28)):

$$Q_0 = \frac{\partial \rho}{\partial \phi} \dot{\phi} \,. \tag{3.7}$$

- Derivative coupling (Eqs. (2.21),(2.29)):

$$Q_0 = -3Hn^2 \frac{\partial f}{\partial n} \dot{\phi} \,. \tag{3.8}$$

The cosmological background dynamics of scalar-fluid theories at the background level were studied in terms of phase-space analysis in [43], for the algebraic coupling case, and in [44] for the derivative coupling case. The results in those two works imply the existence of many new realizations of viable cosmological evolutions in the scalar-fluid theories. Hence, it would be interesting to understand their implications to the structure formation.

3.2 Scalar perturbations

In the present work our main focus is on the behavior of the scalar-fluid cosmological scenarios at the perturbation level. We will first extract the general equations for scalar perturbation, and then use them in order to examine the signatures of these models in the structure formation. We shall continue in a unified notation until the end of this section and then perform a detailed analysis separately for the algebraic couplings and for the derivative couplings.

We will work in the Newtonian gauge, unless otherwise specified. The Newtonian frame is also called the longitudinal gauge, since there the perturbed metric can be written (in Cartesian coordinates) as

$$\mathrm{d}s^2 = -(1 + 2\Phi)\mathrm{d}t^2 + \frac{(1 - 2\Psi)\, a^2(t)}{\left[1 + \frac{1}{4}K\left(x^2 + y^2 + z^2\right)\right]^2} \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \,, \tag{3.9}$$

where Ψ, Φ are functions of all coordinates. Since in our case all the matter sources can be considered as perfect fluids, no anisotropic stresses appear in the considered scalar-fluid models. Hence, the off-diagonal ij-components of the Einstein field equations always read as

$$\partial_i \partial_j \left\{ \left[1 + \frac{1}{4}K\left(x^2 + y^2 + z^2\right)\right] (\Psi - \Phi)\right\} = 0 \,, \tag{3.10}$$

and hence we can immediately deduce that

$$\Phi = \Psi \,, \tag{3.11}$$

as expected in the absence of source imperfections (i.e. off-diagonal pressures). Thus, in what follows we will simplify the equations by setting Φ equal to Ψ.

Let then first determine the matter variables that are going to be perturbed. In scalar-fluid theories the fundamental matter variables of the coupled system are[4] n, ϕ and U^μ (s is neglected due to adiabaticity as we mentioned above) and therefore we should consider the perturbations of these quantities as

$$\phi \mapsto \phi + \delta\phi \,, \qquad n \mapsto n + \delta n \,, \qquad U_\mu \mapsto U_\mu + \delta U_\mu \,, \tag{3.12}$$

where ϕ, n and U_μ are now the background quantities, and

$$\delta U_\mu = (-\Psi, \partial_i v) \,, \tag{3.13}$$

with v the scalar perturbation of the matter fluid velocity[5]. We parametrize the fluctuations in the energy density as $\delta \equiv \delta\rho/\rho$, and for the pressure perturbation we simply write δp. These two quantities have a different relation to δn depending on the perturbation of the general definitions of ρ and p given in Eqs. (2.28) and (2.29). Below we will also give the relation between $\delta\rho$ and δp in detail for the two classes of models studied here.

We are then ready to write down the remaining field equations. There are three independent Einstein equations remaining, since we have already used one to set $\Phi = \Psi$. These can be written as follows, lumping the matter and field fluctuations on the RHS and the geometric fluctuations on the LHS:

$$\left(6\frac{K}{a^2} - \frac{k^2}{a^2} - \kappa^2\rho - \kappa^2 V \right) \Psi - 3H\dot\Psi = \frac{\kappa^2}{2} \left(\rho\delta + \dot\phi\dot{\delta\phi} + V'\delta\phi \right) \,, \tag{3.14}$$

$$\dot\Psi + H\Psi = -\frac{\kappa^2}{2} \left[(\rho + p)\, v - \dot\phi\delta\phi \right] \,, \tag{3.15}$$

$$\ddot\Psi + 4H\dot\Psi + \left(2\dot H + 3H^2 - \frac{K}{a^2} + \frac{1}{2}\kappa^2\dot\phi^2 \right) \Psi = \frac{\kappa^2}{2} \left(\delta p + \dot\phi\dot{\delta\phi} - V'\delta\phi \right) \,. \tag{3.16}$$

Here k is the wavenumber of the fluctuation (where in k-space $\nabla^2 \to -k^2$ with suitable scalar harmonic eigenfunctions that for $K = 0$ reduce to plane waves). To complete the system, we need also the conservation equations for the fluid and the field. We will study these separately in the two cases at hand.

4 Structure formation

In this section we explicitly elaborate the way from the general perturbation equations to the physics of structure formation at the linear regime.

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[4] The actual variable upon which the scalar-fluid action is varied, is J^μ and not U^μ, however according to (2.2) these are related through $J^\mu = \sqrt{-g}\, n U^\mu$, and thus one can always consider U^μ instead of J^μ.

[5] The fact that $\delta U_0 = -\Psi$ comes from perturbing the relation $g^{\mu\nu}U_\mu U_\nu = -1$.

4.1 Algebraic couplings

In this subsection we analyze the scalar perturbations in the case of algebraic couplings [43], characterized by Eqs. (2.20), (2.28) and (3.7). We will need various derivatives of the coupling functions, and it turns out to be useful to define the following auxiliary background quantities (all with the dimensions of $1/M$):

$$
x \equiv \frac{n\rho_{,n\phi}}{(1+w)\,\rho}\,, \quad y \equiv \frac{\rho_{,\phi}}{\rho}\,, \quad z \equiv \sqrt{\frac{\rho_{,\phi\phi}}{\rho}}\,, \quad u \equiv \sqrt{\frac{n\rho_{,n\phi\phi}}{(1+w)\,\rho}}\,, \quad r \equiv \frac{n^2\rho_{,nn\phi}}{(1+w)\rho}\,. \tag{4.1}
$$

Furthermore, we introduce the quantity that is the sound speed square of *the fluid in the rest frame of the field*:

$$
c_s^2 \equiv n\frac{\rho_{,nn}}{\rho_{,n}}\,. \tag{4.2}
$$

If the field is decoupled from the fluid, this coincides with the usual definition of the adiabatic sound speed squared (the background quantity) $c_A^2 = \dot{p}/\dot{\rho}$, and at small scales it coincides with the usual definition of the sound speed squared (perturbation quantity) that is evaluated in the rest frame of the fluid, $\hat{c}_s^2 = \hat{\delta}p/\hat{\delta}\rho$, where the hat indicates that the fluctuations are considered in the comoving matter gauge wherein $\hat{v} = 0$. It turns out that in the limit we will consider, the effective sound speed is indeed given by Eq. (4.2), since at sub-horizon scales the difference between the two rest frames is negligible. However, the reader is warned that fundamentally our definition is slightly different from the conventional one, for purposes of convenience in the present case. At the background the evolution of the pressure is given by (recall that the equation of state is defined by $w = p/\rho$)

$$
\dot{w} = (1+w)\left[3H\left(w - c_s^2\right) + (x-y)\,\dot{\phi}\right]\,, \tag{4.3}
$$

and for the perturbations the total fluid pressure is given as

$$
\frac{\delta p}{\rho} = c_s^2\delta + \left[(1+w)\,x - \left(1+c_s^2\right)y\right]\delta\phi\,, \tag{4.4}
$$

which shows that (4.2) would indeed give the total pressure perturbation of the fluid only in the rest frame of the field $\delta\phi = 0$, and that in general (4.2) differs from $\hat{c}_s^2$.

We can then write the conservation equations for our coupled scalar-fluid system. The Klein-Gordon equation for the fluctuation of the field is

$$
\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left[\frac{k^2}{a^2} + V'' + \rho\left(z^2 - xy\right)\right]\delta\phi + 2\left(\rho y + V'\right)\Psi - 4\dot{\phi}\dot{\Psi} + \rho x\delta = 0\,. \tag{4.5}
$$

The continuity and the Euler equation for the fluctuations in the matter fluid come out respectively as

$$
\dot{\delta} + \left[3H\left(c_s^2 - w\right) + \dot{\phi}\left(y - x\right)\right]\delta - (1+w)\left(\frac{k^2}{a^2}v + 3\dot{\Psi}\right) = y\delta\dot{\phi}
$$
$$
+ \left\{3H\left[\left(1+c_s^2\right)y - (1+w)\,x\right] - \dot{\phi}\left(xy - z^2\right)\right\}\delta\phi\,, \tag{4.6}
$$

and

$$
\dot{v} - \left(3Hc_s^2 - \dot{\phi}x\right)v + \Psi = \left(\frac{c_s^2}{1+w}y - x\right)\delta\phi - \frac{c_s^2}{1+w}\delta\,. \tag{4.7}
$$

This completes the perturbation system for the models defined by Eq. (2.20). Definitely, when we switch off the coupling by setting $x, y, z \to 0$, we recover the usual conservation equations. In the extensively studied case of conformally coupled cold dark matter, corresponding to $\rho = e^{\beta\phi} m_0 n$, we have simply $x = y = z = u = r = \beta$, and the equations simplify to

$$\text{C-coupled CDM}: \quad \begin{cases} \delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(\frac{k^2}{a^2} + V''\right)\delta\phi + 2V'\Psi - 4\dot{\phi}\dot{\Psi} = -\beta\rho\left(\delta + 2\Psi\right), \\ \dot{\delta} - \frac{k^2}{a^2}v - 3\dot{\Psi} = \beta\delta\dot{\phi}, \\ \dot{v} - \beta\dot{\phi}v + \Psi = -\beta\delta\phi. \end{cases} \tag{4.8}$$

Obviously, allowing ρ to be a general function of both the scalar field and the number density of matter particles results in a richer structure of the equations. We proceed now to explore their cosmological implications.

4.1.1 The quasi-static limit

Cosmological large scale structure can be probed most efficiently at sub-horizon scales, since beyond that we face problems related to both systematic difficulties and cosmic variance. At the subhorizon scales, we can considerably simplify the system of perturbation equations by considering taking advantage of the so called quasi-static limit, that will be explained in the process of exploiting it below.

To begin, we combine equations (3.14) and (3.15) to obtain the generalized Poisson equation in the quasi-static limit:

$$\left[6\frac{K}{a^2} - \left(\frac{k}{a}\right)^2 + \frac{\kappa^2}{2}\left(\rho + V + \dot{\phi}^2\right)\right]\Psi = \frac{\kappa^2}{2}\left[\rho\hat{\delta} + \left(V' + 3H\dot{\phi}\right)\delta\phi + \dot{\phi}\delta\dot{\phi}\right]$$

$$\Rightarrow \quad -\left(\frac{k}{a}\right)^2\Psi = \frac{\kappa^2}{2}\rho\hat{\delta}. \tag{4.9}$$

In the first line we have eliminated the $\dot{\Psi}$ and at the same time obtained the density perturbation in the comoving matter gauge, $\hat{\delta} = \delta - 3H(1+w)v$, which is the quantity we are interested in[6], as we presumably are comoving with matter, and thus this is the gauge corresponding to physical observables such as the matter power spectrum, $\sim \langle\hat{\delta}^2\rangle$. Let us now discuss in detail how we can justify, within our approximation, to drop each of the additional terms that appear in the first line of the equation above. Firstly, at the sub-horizon scales we consider, $k^2 \gg K$, as the spatial curvature scale of the universe is

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[6]There is also a subtlety involved here. In fact we could argue that the quasi-static limit of (3.14) would directly yield the Poisson equation (4.9) with $\hat{\delta}$ replaced by the Newtonian gauge δ. However, in the presence of non-negligible pressures, the velocity fluctuations and thus the difference in the gauges can be significant, and though not naively apparent from (3.14) alone, the quasi-static limit of the full system would be inconsistent if we removed the hat from (4.9). Care has to be taken with the derivatives as it turns out that in fact $\ddot{\Psi} \simeq \kappa^2 c_s^2 \rho\hat{\delta}/2$ in the subhorizon approximation (and in this sense the quasi-static approximation is not strictly valid), though still $\dot{\Psi} \sim \mathcal{O}(H\Psi)$ and thus we are able to close the system consistently. In particular, the approximated system of equations we eventually obtain, described by the result (4.11), satisfies all the field equations and the Bianchi identities.

constrained by observations to be at least beyond the present horizon. Secondly, since $\kappa^2 \rho \lesssim H^2$ as well as $\kappa^2 V, \kappa^2 \dot{\phi}^2 \lesssim H^2$, and, by definition of the subhorizon limit, we consider scales for which $k^2 \gg a^2 H^2$, the gradient term in the LHS square brackets clearly dominates over the other terms. Similarly, in the RHS the matter perturbation (term $\sim \hat{\delta}$) will dominate over the field fluctuation (terms $\sim \delta\phi, \delta\dot{\phi}$). At small scales the field may perform rapidly small oscillations, and thus we only need to take into account its averaged amplitude. This we will soon see to be given by the coupling to matter, in such a way that the field fluctuations are indeed suppressed in comparison to the matter overdensities, $\kappa \delta\phi \sim (H/k)^2 \delta$.

Therefore, using (4.9), we can relate the gravitational potential Ψ directly to the matter overdensity δ, and turn the evolution equation for Ψ, the space-space component (3.16) of the field equations, into an evolution equation for δ. However, in order to close the system we also need to express the scalar field perturbation (and its derivative) in terms of δ. This can be accomplished by using the quasi-static limit of the Klein-Gordon equation (4.5), namely

$$-\left(\frac{k}{a}\right)^2 \delta\phi = x\rho\delta \,. \tag{4.10}$$

We arrived at this simple relation by repeating the arguments above that we used to justify Eq. (4.9). The time-derivative of the gravitational potential is of the order of magnitude $\dot{\Psi} \sim H\Psi$, since, as in general also for other fluctuations, the time-scale of the evolution is given by the cosmological expansion rate: $d/dt \sim H$. We have then everything needed to close the system of equations. In equation (3.16), where the pressure perturbation is now given by (4.4), we first substitute Ψ and its derivatives using (4.9) and $\delta\phi$ and its derivatives using (4.10), and then we elaborate the resulting equation using the background formulas (3.2)-(3.6) and (4.1)-(4.3), obtaining

$$\ddot{\hat{\delta}} + \left[\left(2 - 6w + 3c_s^2\right) H + 2y\dot{\phi}\right] \dot{\hat{\delta}} + c_s^2 \left(\frac{k}{a}\right)^2 \hat{\delta} = \left(C_0 + C_1 H\dot{\phi} + C_2 \dot{\phi}^2\right) \hat{\delta} \,, \tag{4.11}$$

where

$$C_0 = \frac{\kappa^2}{2} \left[\left(1 - 3w\right)\left(1 + w\right)\rho + \left(4c_s^2 - 3w\right)\dot{\phi}^2\right] + \left(15w - 9c_s^2\right) H^2 + \left(2 + 3w\right)\frac{K}{3a^2}$$
$$+ \left[\left(1 + w\right)x^2 - \left(1 + c_s^2\right)xy + y^2\right]\rho - \left(x - y\right)V' \,, \tag{4.12}$$

$$C_1 = \left(5 + 6c_s^2 + 3w\right)x - \left(5 + 3c_s^2\right)y - 3r \,, \tag{4.13}$$

$$C_2 = u^2 - x^2 - xy - 4y^2 \,. \tag{4.14}$$

This result is applied in Appendix B.1 to analyze a specific class of Lagrangians.

We close this subsection by checking some special limits of this equation. In the following let us set $K = 0$ for simplicity. Firstly, if we switch off the scalar field, we recover the *exact* evolution equation for overdensities in the uncoupled matter distribution:

$$\text{Matter only:} \quad \ddot{\hat{\delta}} + \left(2 - 6w + 3c_s^2\right) H\dot{\hat{\delta}} + \frac{3}{2}\left[1 + (8 - 3w)w - 6c_s^2\right] H^2\hat{\delta} = -\hat{c}_s^2 \left(\frac{k}{a}\right)^2 \hat{\delta} \,. \tag{4.15}$$

The hatted sound speed should be used in the entropic case (though not considered here). The mass-varying cold dark matter, described by equations (4.8), results in

$$\text{C-coupled CDM:} \quad \ddot{\hat{\delta}} + \left(2H - \beta\dot{\phi} \right) \dot{\hat{\delta}} = \left(\frac{\kappa^2}{2} + \beta^2 \right) \rho\hat{\delta} \,. \tag{4.16}$$

In addition to the friction term on the LHS, on the RHS the coupling always adds to the effective Newton's constant: the fifth force between CDM particles that is mediated by the scalar field is always attractive[7]. Both of these effects are significant in the perturbation evolution, and generically modify the matter power spectrum too much to comply with observations, in cases when the coupling β is large enough to qualitatively change the background evolution: this is a basic difficulty in addressing the coincidence problem within the standard coupled quintessence models featuring a canonical scalar field coupled to cold dark matter [32].

In general, however, modifications arising in the form of the effective sound speed term in (4.11) are still more disastrous to the matter power spectrum due to their k^2-enhanced influence at smaller scales. Therefore, we can exclude any appreciable nonlinear dependence of the fluid Lagrangian upon the number density. We will return to discuss this result in Section 5, after noting that it applies also for the derivatively coupled models in section 4.2.

4.2 Derivative couplings

Let us now consider the case of derivative couplings [44], characterized by Eqs. (2.21), (2.29) and (3.8). In order to simplify the notation in what follows we define

$$x = n^2 \frac{\partial^2 f}{\partial\phi\partial n} \,, \qquad y = n^2 \frac{\partial f}{\partial n} \,, \qquad z = n^3 \frac{\partial^2 f}{\partial n^2} \,. \tag{4.17}$$

From (2.21), we now obtain for the pressure at the background and at the perturbation levels

$$p = n\rho_{,n} - \rho - y\dot{\phi} \,, \tag{4.18}$$

$$\delta p = \left(c_s^2\rho - \frac{2y+z}{1+w_0}\dot{\phi} \right) \delta + y\dot{\phi}\Psi - x\dot{\phi}\delta\phi - y\dot{\delta\phi} \,, \tag{4.19}$$

respectively, where $\delta = \delta\rho/\rho$, w_0 (the "bare" EoS parameter) is given by

$$w_0 \equiv n\frac{\rho_{,n}}{\rho} - 1 = w + \frac{y\dot{\phi}}{\rho} \,, \tag{4.20}$$

and c_s^2 is defined exactly as in Eq. (4.2), although now ρ does not depend on the scalar field. In analogy with w_0, c_s^2 corresponds now to the "bare" sound speed and we remind that it is not the "full" $\hat{c}_s^2$ as explained after (4.2). The evolution of the equation of state is given by

$$\dot{w} = 3H \left(1 + w_0 \right) \left(w - c_s^2 \right) - \frac{1}{\rho} \left\{ x\dot{\phi}^2 + 3H \left[(2y+z)\,\dot{\phi} - y^2 \right] V'y \right\} \,. \tag{4.21}$$

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[7]This result holds with generalized "disformal" couplings too [73], though could be avoided at least with a (pathological) phantom scalar field [74].

It is a curious property of the coupling that it adds only a contribution to the effective pressure. This allows for example to construct a de Sitter expansion dominated by the canonical kinetic energy of the scalar field: the scalar can have a constant time-derivative due to the coupling [44].

The perturbed scalar field equation reads

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \left(\frac{k^2}{a^2} - 3Hx + V''\right)\delta\phi - \left(3Hy - 2V'\right)\Psi - \left(4\dot{\phi} - 3y\right)\dot{\Psi} = 3H\left(\frac{2y+z}{1+w_0}\right)\delta - y\frac{k^2}{a^2}v.$$
(4.22)

In contrast to the algebraically coupled models, here the scalar fluctuations are sourced also by the velocity perturbation in the matter field. The continuity equation is simply

$$\dot{\delta} + 3(c_s^2 - w_0)H\delta - (1 + w_0)\left(\frac{k^2}{a^2}v + 3\dot{\Psi}\right) = 0,$$
(4.23)

and thus retains its usual form unaffected by the coupling. This can be seen from the covariant equation (2.31), that for the derivative couplings (3.8) reduces to $U^\mu\nabla_\mu\rho + (1 + w_0)\rho\nabla_\mu U^\mu = 0$, where the scalar-field dependence has vanished identically because the "fifth force" is orthogonal to the fluid flow as shown in [44]. In the Euler equation, which is (2.32) adapted to cosmology, we however obtain nontrivial coupling terms as the spatial components are subject to the interaction:

$$(1 + w)\rho\dot{v} + \left\{3H\left[(1 + w_0)c_s^2\rho - (2y + z)\dot{\phi} - y^2\right] + x\dot{\phi}^2 - yV'\right\}v$$
$$+ \left(c_s^2\rho - \frac{2y+z}{1+w_0}\dot{\phi}\right)\delta + (1 + w_0)\rho\Psi = \left(3Hy + x\dot{\phi}\right)\delta\phi + y\dot{\delta\phi}.$$ (4.24)

Note that in the equations z appears only in the combination $2y + z$.

4.2.1 The quasi-static limit

We can now consider the derivatively coupled models in the quasi-static approximation. Since our focus is upon the novel effects arising from the gradient-type interaction, we consider the matter source to be cold, $\rho \propto n$ by itself for simplicity, that is we set $w_0 = c_s^2 = 0$.

In addition, we restrict the analysis to the special class of couplings for which $f \propto 1/n$. The reason is that in the first place the quasi-static approximation would not be justified in the presence of any nonlinear function n^2f; such a term would contribute an effective pressure as a source in complete analogy with a nonlinear $\rho(n)$ term, the difference now being that the pressure term due to the coupling f appears with the gradient-type prefactor and it is thus strongly enhanced at small scales. The two effects combined, extra pressure and extra gradient, would probably lead to disaster for structure formation at subhorizon scales (unless the coupling is tuned to be extremely tiny), and since then the scalar field would cluster together with matter also inside the horizon, the quasi-static approximation would not be valid for these models[8].

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[8]In fact, if we follow the steps of subsection (4.1.1) in such a case, we would find that the solution for the $\delta\phi$-derivatives, in addition to derivatives of Ψ, are of the order of δ, and then we could not close the system consistently (one could though reduce it to two coupled second-order equations without the quasi-static approximation) - see also the comments in the previous case in the footnote 6 of section 4.1.1.

Hence, in what follows we specialize to cold dark matter with the derivative coupling $z = -y/2$ and $x = y'$, where a prime denotes differentiation with respect to ϕ and $y(\phi)$ has to be taken as a general function of the scalar field. This simplifies the system considerably. Similarly to the previous case, the time-time component (3.14) of the field equations reduces to the usual Poisson equation in the quasi-static limit:

$$-\left(\frac{k}{a}\right)^2 \Psi = \frac{\kappa^2}{2}\rho\hat{\delta}\,. \tag{4.25}$$

Thus, we can again relate the gravitational potential Ψ directly to the matter overdensity δ, and turn the evolution equation for Ψ, the space-space component (3.16) of the field equations, into an evolution equation for δ. However, to close the system we also need to express the field perturbation (and its derivative) in terms of δ (and its derivatives):

$$\delta\phi = -yv = -y\left(\frac{a}{k}\right)^2\hat{\delta}\,, \tag{4.26}$$

where we used the appropriate limit of Eq. (4.22) for the first, and of Eq. (4.23) for the second equality. Then, after some algebra we arrive at the following closed form equation for $\hat{\delta}$:

$$\ddot{\hat{\delta}} + \left[2H + \frac{V'y + y'\left(2y - \dot{\phi}\right)\dot{\phi}}{\rho + y\left(y - \dot{\phi}\right)}\right]\dot{\hat{\delta}} = \frac{\kappa^2}{2}\rho\left[1 + \frac{y\left(y - \dot{\phi}\right)}{\rho}\right]^{-1}\hat{\delta}\,. \tag{4.27}$$

We stress that now the effective gravitational constant can receive positive as well as negative contribution from the coupling, whereas in the most standard cases the fifth force is always attractive, as is clear from (4.16).

In appendix B.2 we briefly apply the above results to a concrete example, considering the models specified by $f = -\xi H_0/(\kappa n)$ (so for which $y = \xi H_0/\kappa$ is just a constant), whose background phase-space analysis was performed in [44].

5 Discussion and outlook

In this work we studied the formation of cosmological large-scale structure in the new class of Scalar-Fluid theories. These theories are based on an extension of the Lagrangian formalism of effective perfect fluids, that allows for new scalar-field-dependent terms in the fluid Lagrangian, introducing non-minimal interactions of the perfect fluid and the scalar field. We explored the cosmological implications of the new couplings of both the algebraic type (i.e. when the fluid energy density may depend non-trivially both upon the number density of particles and the scalar field, thus generalizing the usual Yukawa-type coupling $\rho = m(\phi)n \rightarrow \rho(\phi, n)$) and of the derivative type (i.e. where the interaction is between the four-gradient of the field and the four-velocity of the fluid, contributing only to the effective pressure term, $p \rightarrow p - n^2\frac{\partial f(n,\phi)}{\partial n}U^\mu\phi_\mu$).

The cosmological equations were derived up to linear order in perturbations, and furthermore the perturbation system was reduced to a simple second order closed evolution

(master) equation for the observable matter structures $\hat{\delta}$ at the relevant subhorizon limit, where we were able to employ (partially) the Newtonian quasi-static approximation. The main results of the investigation were given as Eq. (4.11) for the algebraic case and as Eq. (4.27) for the derivative case. Compared to the standard result of Eq. (4.15), we observe three different modifications in the matter perturbation evolution:

- *Additional friction terms.* In addition to the usual Hubble-friction that slows down the clumping of matter due to the expansion of the universe, the couplings typically introduce additional friction terms that are given by the evolution of the scalar field.

- *Modifications of the effective Newton's constant.* Matter falls into the potential wells induced by matter, but the rate at which this occurs can be modified by the coupling. This can be interpreted as an extra force between matter particles mediated by the scalar field, and described as a varying amplitude of the effective Newton's constant.

- *Effective pressure perturbations.* Any nonlinear dependence on the number density of the fluid particles in the Lagrangian means nontrivial self-interactions. These will introduce pressures in the fluid, thus implying propagation of sound waves. In the matter fluctuation equations these appear as gradient-like terms that become increasingly important at small scales.

Given any specific model, i.e. given any specific Scalar-Fluid Lagrangian, these effects can be straightforwardly determined from equation (4.11) or equation (4.27), which can then be conveniently integrated to obtain the growth rate or the matter power spectrum as a function of the redshift; see appendix B.

In the context of interacting dark matter - dark energy cosmologies, the sound speed term is the most dangerous for the viability of the models. Such terms have been known to appear in coupled three-form models [24] and in unified dark matter scenarios (such as the Chaplygin gas or Cardassian expansion models [67–69]), and it is the culprit which makes these constructions ruled out - or rather constrained to a tiny parameter region where they are completely indistinguishable from the standard ΛCDM model, at least by any realistic background measurement [75, 76]. For some modified gravity models that generate such effective pressure perturbations as source for matter fluctuations, the perturbation constraints have been also shown to tighten the viability bounds ensuing from background expansion alone by several orders of magnitude [77, 78]. We therefore know that any nonlinearities in matter density, either in the minimally or in the coupled matter terms in the Lagrangian, could not have any appreciable influence on the background evolution[9].

Even restricting to models which exhibit no effective pressure gradients at the perturbation level, we have at hand new classes of potentially viable interacting dark sector cosmologies. The constraints we can impose on them by a detailed comparison of the the-

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[9]A possible caveat is that including both non-standard properties for dark matter and non-trivial interactions, the new effects could in principle cancel each other in order to eliminate the dangerous sound speed term [79]. This would however appear to require a very fine-tuned conspiracy.

ory predictions and the cosmological data remain to be studied. The present work provides the tools to carry out such analysis up to linear order in cosmological perturbations.

Let us end with the remark that there might be interesting scalar-fluid theories besides the two classes focused upon here. In particular, as discussed in appendix C, high energy physics considerations naturally suggest more general forms of derivative couplings than the linear one studied here, and in addition one could contemplate also on modifying the Lagrangian constraints included in the theory. In this way one could change for example explicitly the number conservation of particles, implying scalar-field dependent particle creation, which could be phenomenologically useful e.g. for the reheating process after inflation. Furthermore, including dynamics for entropic degrees of freedom and extending to imperfect fluids, are amongst other possible generalizations to consider in the future.

Acknowledgments

A   Alternative representation

In this appendix we will present the cosmological perturbation equations in the notation originally adopted in [43, 44], where the fluid Lagrangian is separated into its minimal and interacting part. Action (2.17) is written in this notation as

$$\mathcal{S} = \int d^4x \Big[\mathcal{L}_{\text{grav}} + \mathcal{L}_\phi + \mathcal{L}_M^{(minimal)} + \mathcal{L}_{\text{int}} \Big] , \tag{A.1}$$

where $\mathcal{L}_{\text{grav}}$, $\mathcal{L}_\phi$ and $\mathcal{L}_M^{(minimal)}$ are respectively defined by Eqs. (2.18), (2.19) and (2.1), while the interacting Lagrangian assumes the forms

$$\mathcal{L}_{\text{int}} = -f(n, s, \phi) \qquad \text{for algebraic couplings} , \tag{A.2}$$

$$\mathcal{L}_{\text{int}} = f(n, s, \phi) J^\mu \partial_\mu \phi \quad \text{for derivative couplings} . \tag{A.3}$$

Note that the meaning of $f(n, s, \phi)$ is different for the two types of models and one must not be confused by this small abuse of notation. We will refer to action (2.17) as the "implicit" representation, while action (A.1) will be called the "explicit" representation. Such names derive from the fact that in (2.17) the scalar-fluid coupling implicitly appears within the fluid Lagrangian, while in (A.1) it explicitly appears as a separate term.

The main difference between the two representations is given by the definitions of the fluid energy density and pressure. In the implicit representation we have the expressions given by Eqs. (2.28) and (2.29), while in the explicit representation the energy density and pressure are solely given by the minimal fluid Lagrangian as

$$\rho = \rho(n, s) \quad \text{and} \quad p = n \frac{\partial \rho(n, s)}{\partial n} - \rho(n, s) . \tag{A.4}$$

They thus never contain any dependence on the scalar field, which will then always appear in separate terms within the field equations. The relation between the energy density and

pressure in the two representations coincides with the "tilde" transformations considered in [43, 44] and then generalized to phenomenological models of interacting dark energy in [58].

In this appendix we will not present the covariant field equations as well as the background cosmological equations in the explicit representation since these have already been given in [43, 44]. We will only provide the corresponding perturbation equations following the same notation and conventions we adopted in the main text. Again because of the absence of anisotropic stresses the constraint $\Psi = \Phi$ will apply, allowing us to simplify the equations.

For the algebraically coupled models (A.2) the perturbed equations in Newtonian gauge are then

$$-3H\dot{\Psi} + \left[6\frac{K}{a^2} - \frac{k^2}{a^2} - \kappa^2\left(\rho + V + f\right)\right]\Psi - \frac{\kappa^2}{2}\left(\frac{\partial f}{\partial n}\frac{n}{\rho + p} + 1\right)\delta\rho$$
$$-\frac{\kappa^2}{2}\left(\frac{\partial f}{\partial\phi} + V'\right)\delta\phi - \frac{\kappa^2}{2}\dot{\phi}\dot{\delta\phi} = 0\,, \qquad \text{(A.5)}$$

$$\dot{\Psi} + H\Psi + \frac{\kappa^2}{2}\left(\rho + p + n\frac{\partial f}{\partial n}\right)v - \frac{\kappa^2}{2}\dot{\phi}\delta\phi = 0\,, \qquad \text{(A.6)}$$

$$-\frac{\kappa^2}{2}\delta p - \frac{\kappa^2}{2}\frac{\partial^2 f}{\partial n^2}\frac{n^2}{\rho + p}\delta\rho + \frac{\kappa^2}{2}\left(\frac{\partial f}{\partial\phi} - n\frac{\partial^2 f}{\partial n\partial\phi} + V'\right)\delta\phi - \frac{\kappa^2}{2}\dot{\phi}\dot{\delta\phi}$$
$$+ \left(2\dot{H} + 3H^2 + \frac{\kappa^2}{2}\dot{\phi}^2 - \frac{K}{a^2}\right)\Psi + 4H\dot{\Psi} + \ddot{\Psi} = 0\,, \qquad \text{(A.7)}$$

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \left(\frac{k^2}{a^2} + \frac{\partial^2 f}{\partial\phi^2} + V''\right)\delta\phi + \frac{\partial^2 f}{\partial n\partial\phi}\frac{n}{\rho + p}\delta\rho - 2\left(\ddot{\phi} + 3H\dot{\phi}\right)\Psi - 4\dot{\phi}\dot{\Psi} = 0\,, \quad \text{(A.8)}$$

$$\dot{\delta\rho} + 3H\left(\delta\rho + \delta p\right) - (\rho + p)\frac{k^2}{a^2}v - 3(\rho + p)\dot{\Psi} = 0\,, \qquad \text{(A.9)}$$

$$\left(\frac{\partial f}{\partial n} + \frac{\partial\rho}{\partial n}\right)\dot{v} + \left[\dot{\phi}\frac{\partial^2 f}{\partial n\partial\phi} - 3Hn\left(\frac{\partial^2 f}{\partial n^2} + \frac{\partial^2\rho}{\partial n^2}\right)\right]v + \frac{\partial^2 f}{\partial n^2}\frac{n}{\rho + p}\delta\rho + \frac{\partial^2 f}{\partial n\partial\phi}\delta\phi$$
$$+ \frac{1}{n}\delta p + \left(\frac{\partial f}{\partial n} + \frac{\partial\rho}{\partial n}\right)\Psi = 0\,, \quad \text{(A.10)}$$

where $\delta\rho$ and δp are related to δn through the perturbation of Eqs. (A.4), and ρ, p, ϕ and n now denote background quantities. Eqs. (A.5)–(A.7) arise from the perturbed Einstein field equations, while Eq. (A.8) is derived from the scalar-field equation, and Eqs. (A.9) and (A.10) correspond to the perturbed matter conservation equations. Note how the interacting function f explicitly appears in various terms, when these equations are compared to Eqs. (3.14)–(3.16) and (4.5)–(4.7).

For the derivatively coupled models (A.3) the perturbed equations in Newtonian gauge are instead

$$\left(6\frac{K}{a^2} - \frac{k^2}{a^2} - \kappa^2\rho - \kappa^2 V\right)\Psi - 3H\dot{\Psi} - \frac{\kappa^2}{2}\delta\rho - \frac{\kappa^2}{2}V'\delta\phi - \frac{\kappa^2}{2}\dot{\phi}\dot{\delta\phi} = 0\,, \qquad \text{(A.11)}$$

$$\dot{\Psi} + H\Psi + \frac{\kappa^2}{2}\left(\rho + p - n^2\dot{\phi}\frac{\partial f}{\partial n}\right)v - \frac{\kappa^2}{2}\dot{\phi}\delta\phi = 0\,, \tag{A.12}$$

$$\ddot{\Psi} + 4H\dot{\Psi} + \left[-\frac{K}{a^2} + 3H^2 + 2\dot{H} + \frac{\kappa^2}{2}\dot{\phi}^2 - \frac{\kappa^2}{2}n^2\dot{\phi}\frac{\partial f}{\partial n}\right]\Psi + \frac{\kappa^2}{2}\dot{\phi}\left(2\frac{\partial f}{\partial n} + n\frac{\partial^2 f}{\partial n^2}\right)\frac{n^2}{\rho + p}\delta\rho$$
$$-\frac{\kappa^2}{2}\delta p + \frac{\kappa^2}{2}\left(n^2\dot{\phi}\frac{\partial^2 f}{\partial\phi\partial n} + V'\right)\delta\phi + \frac{\kappa^2}{2}\left(n^2\frac{\partial f}{\partial n} - \dot{\phi}\right)\dot{\delta\phi} = 0\,, \tag{A.13}$$

$$3H\left(2\frac{\partial f}{\partial n} + n\frac{\partial^2 f}{\partial n^2}\right)\frac{n^2}{\rho + p}\delta\rho + \left(2\ddot{\phi} + 6H\dot{\phi} - 3Hn^2\frac{\partial f}{\partial n}\right)\Psi + \left(4\dot{\phi} - 3n^2\frac{\partial f}{\partial n}\right)\dot{\Psi}$$
$$-\frac{k^2}{a^2}n^2\frac{\partial f}{\partial n}v + \left(-\frac{k^2}{a^2} + 3Hn^2\frac{\partial^2 f}{\partial\phi\partial n} - V''\right)\delta\phi - 3H\dot{\delta\phi} - \ddot{\delta\phi} = 0\,, \tag{A.14}$$

$$\frac{1}{\rho + p}\left(\dot{\delta\rho} + 3H\delta p + 3H\delta\rho\right) - \frac{k^2}{a^2}v - 3\dot{\Psi} = 0\,, \tag{A.15}$$

$$\left(\frac{\partial\rho}{\partial n} - n\dot{\phi}\frac{\partial f}{\partial n}\right)\dot{v} + n\left[3H\dot{\phi}\left(\frac{\partial f}{\partial n} + n\frac{\partial^2 f}{\partial n^2}\right) - 3H\frac{\partial^2\rho}{\partial n^2} - \ddot{\phi}\frac{\partial f}{\partial n} - \dot{\phi}^2\frac{\partial^2 f}{\partial\phi\partial n}\right]v + \frac{1}{n}\delta p$$
$$-\dot{\phi}\left(2\frac{\partial f}{\partial n} + n\frac{\partial^2 f}{\partial n^2}\right)\frac{n}{\rho + p}\delta\rho + \frac{\partial\rho}{\partial n}\Psi - n\left(3H\frac{\partial f}{\partial n} + \dot{\phi}\frac{\partial^2 f}{\partial\phi\partial n}\right)\delta\phi - n\frac{\partial f}{\partial n}\dot{\delta\phi} = 0\,. \tag{A.16}$$

Note that Eqs. (A.11) and (A.15) are not modified by the coupling. This happens because all terms arising from the interactions are orthogonal to the fluid flow, as already pointed out in [44]. Again the differences between Eqs. (A.11)–(A.16) and Eqs. (3.14)–(3.16) and (4.22)–(4.24) are in the explicit terms depending on the coupling, though for the models (A.3) the definitions of ρ in the implicit and explicit representations coincide.

B Specific solutions

In this appendix we make some first steps towards a numerical elaboration of structure formation in the coupled models. We sketch an approach to implement background phase-space variables in order to conveniently solve the main equations (4.11) and (4.27) for the algebraically and derivatively coupled models respectively, and on the way we obtain some explicit solutions in a couple of special cases where the background admits an analytic (fixed point) solution. The systematic analysis of the full cosmological solutions in realistic models and their confrontation with the data is left for a future project.

B.1 Algebraic couplings

Consider the model specified by

$$\rho(n,\phi) = An^{1+w_0} + Bn^{\alpha(1+w_0)}e^{-\beta\kappa\phi}\,, \quad V(\phi) = V_0 e^{-\lambda\kappa\phi}\,. \tag{B.1}$$

This is equivalent to the model proposed in [43], where in addition to the "bare" energy density described by the equation of state w_0, we have a coupling term proportional to the

power α of the "bare" energy density and the exponent of the field with slope β. We can parametrize the background evolution in terms of the phase-space variables[10] σ, X, Y and Z, defined as

$$\sigma^2 = \frac{\kappa^2 A n^{1+w_0}}{3H^2}, \quad X^2 = \frac{\kappa^2 \dot{\phi}^2}{6H^2}, \quad Y^2 = \frac{\kappa^2 V}{3H^2}, \quad Z = 1 - \sigma^2 - X^2 - Y^2. \tag{B.2}$$

These variables are of course dynamical functions in general. Moreover, the characteristics of the coupled fluid, both at the background and perturbation levels, should be in general considered as functions of time, namely

$$w = \frac{w_0 \sigma^2 + [\alpha(1+w_0) - 1] Z}{\sigma^2 + Z}, \quad c_s^2 = \frac{w_0 \sigma^2 + \alpha[\alpha(1+w_0) - 1] Z}{\sigma^2 + Z}. \tag{B.3}$$

The derivative terms defined in (4.1), that are relevant for the perturbation evolution, for the coupling case (B.1) become

$$x = -\kappa\beta Z \frac{\alpha(1+w_0)}{(1+w)(\sigma^2 + Z)}, \quad y = -\frac{\kappa\beta Z}{\sigma^2 + Z}, \quad z = \kappa\beta\sqrt{\frac{Z}{\sigma^2 + Z}},$$

$$u = \kappa\beta\sqrt{\frac{Z\alpha(1+w_0)}{(1+w)(\sigma^2 + Z)}}, \quad r = -\beta\kappa\frac{Z\alpha(1+w_0)[\alpha(1+w_0) - 1]}{(1+w)(\sigma^2 + Z)}. \tag{B.4}$$

Hence, we can rewrite the evolution equation (4.11) completely in terms of the new variables (B.2), using (B.3),(B.4) above. This can be useful when elaborating the cosmological scenario numerically in terms of the background phase-space variables, but at this stage it might be not clear what we have gained.

The benefit is obvious at the fixed points of the background. At these points the phase-space variables are constants, given by the four parameters of the model $(\alpha, \beta, w_0, \lambda)$, and H scales as

$$H = \frac{2}{3(1 + w_{eff})(t - t_0)}, \quad w_{eff} = w_0 - (w_0 - 1)X^2 - (1 + w_0)Y^2 + (1 + w_0)(\alpha - 1)Z, \tag{B.5}$$

with t_0 an intergration constant. Therefore we can reduce the evolution equation (4.11) to the form

$$\ddot{\hat{\delta}} + \frac{c_1}{t - t_0}\dot{\hat{\delta}} + \frac{c_2}{(t - t_0)^2}\hat{\delta} = k^2 c_3 \left(\frac{\kappa}{t - t_0}\right)^{\frac{4}{3(1 + w_{eff})}} \hat{\delta}, \tag{B.6}$$

where each of the three dimensionless coefficients $c_i = c_i(\alpha, \beta, w_0, \lambda)$ is a constant given by the corresponding background fixed-point solution. Several different types of fixed points have been found for the model (B.1) and their stability properties have been analyzed in [43]. Given any of these solutions, one can immediately apply (B.6) and easily solve it to obtain the growth rate at a given scale as a function of the redshift. In general though, these expressions are rather complicated functions, and for simplicity we will not write them explicitly.

---

[10]Denoted σ, x, y and z, respectively, in Ref. [43]. Here we use capital letters to avoid confusion with the definitions (4.1). We also used a slightly different notation from [43] in (B.1), where the parameters A, B and V_0 are included in order to acquire the correct dimensions.

Let us only consider explicitly a special case for the sake of concreteness. From (B.3) we can confirm that the effective sound speed vanishes identically only if $w_0 = 0$ and $\alpha = 1$ (excluding the trivial case $\alpha = 0$). This is of course what was expected as it is then that the Lagrangian contains only linear dependence upon the matter number density. Specializing to this case, we are left with the model parameters β and λ. Actually, we have then reduced the model to a "conformally coupled" CDM, where $\rho = m(\phi)n$ with now a non-trivial function $m(\phi) = 1 + (B/A)e^{-\beta\kappa\phi}$ (cf. (4.8),(4.16)). There is a well-known matter-scaling fixed point for the exponential quintessence for which $X = Y = \sqrt{3/2}/\lambda$ and $Z = 0$. For this fixed point our equation (B.6) reduces to (setting $t_0 = 0$ for convenience)

$$\ddot{\hat{\delta}} + \frac{4}{3t}\dot{\hat{\delta}} = \frac{2}{3t^2}\left(1 - \frac{3}{\lambda^2}\right)\hat{\delta}\,, \tag{B.7}$$

which is easily solved to yield the growth rate $d\log\hat{\delta}/d\log a = \frac{1}{4}\left(\sqrt{5^2 - 72/\lambda^2} - 1\right)$. In the limit $\lambda \to \infty$ this reduces to the usual matter dominated result $d\log\hat{\delta}/d\log a \to 1$.

To look at another fixed point for which the coupling is significant too, let us consider the exact background solution $X = \sqrt{\frac{3}{2}}\frac{1}{\lambda-\beta}$, $Y = \sqrt{\frac{3+2\beta(\beta-\lambda)}{2}}$, $Z = \frac{\lambda(\lambda-\beta)-3}{(\beta-\lambda)^2}$, namely the conditionally stable fixed point F of [43]. At this fixed point, the equation (B.6) becomes

$$\ddot{\hat{\delta}} + \frac{4\,(4\beta - \lambda)}{3\lambda t}\dot{\hat{\delta}} = -\frac{2\left[2\beta^3\lambda - 2\beta^2\left(\lambda^2 - 12\right) + \beta\lambda - \lambda^2 + 3\right]}{3\lambda^2 t^2}\hat{\delta}\,. \tag{B.8}$$

It is again easy to solve this equation analytically. Now the resulting growth rate is however a more complicated function of the two parameters (the slopes of the coupling and the potential exponential functions). In order to present the results in a more transparent way, in Fig. 1 we plot the resulting growth-rate function as a function of the parameter β, for four choices of the parameter λ (note that physically the fixed point does not exist for all parameter values displayed in the figure, namely it exists only for $\beta(\beta - \lambda) > -3/2$ [43]).

B.2 Derivative couplings

Here we adopt the model specified by

$$f = -\xi H_0/(\kappa n), \tag{B.9}$$

whose background phase space analysis was performed in [44] for the case with an exponential potential. In analogy to our computations of the above subsection, where we used the previous background solutions for the algebraically coupled models, we can now employ the exact fixed-points solutions from [44] for the derivatively coupled models. Let us first look at the matter dominated fixed point $\rho \gg \dot{\phi}^2, V$. At this fixed point Eq. (4.27) becomes

$$\ddot{\hat{\delta}} + \frac{4}{3t}\dot{\hat{\delta}} = \frac{8}{12t^2 + 9\xi^2 H_0^2 t^4}\hat{\delta}\,. \tag{B.10}$$
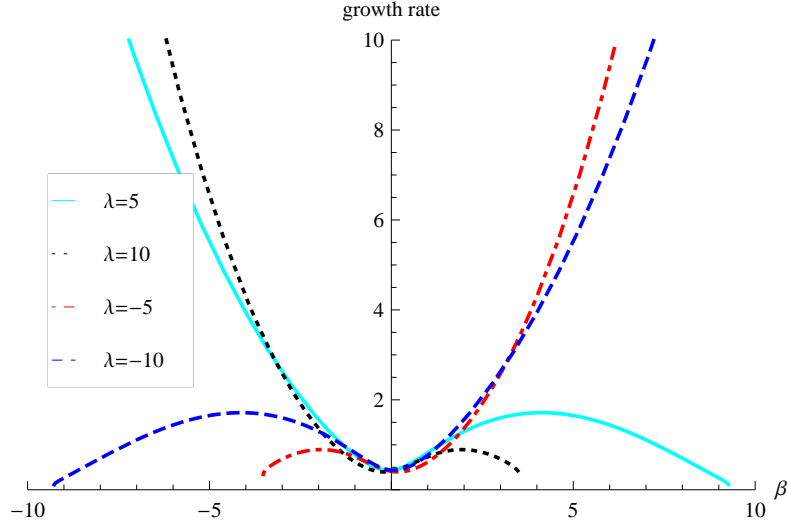
Figure 1. *The growth rate for matter densities for the model (B.1) with $\alpha = 1$, at the scalar-field dominated background solution (fixed point F of [43]). The x-axis is the coupling parameter β and we plot the growing solution for four choices of the potential parameter λ as indicated in the legend.*

An analytic solution exists (with A and B dimensionless integration constants), namely:

$$
\hat{\delta} = \frac{A}{H_0 t} {}_2F_1\left[-\frac{1}{2}, -\frac{1}{3}; \frac{1}{6}, -\frac{3}{4}\xi^2 H_0^2 t^2\right] + B H_0^{\frac{2}{3}} t^{\frac{2}{3}} {}_2F_1\left[\frac{1}{3}, \frac{1}{2}; \frac{11}{6}, -\frac{3}{4}\xi^2 H_0^2 t^2\right]
$$

$$
\Rightarrow \sim t^{-\frac{1}{6}} J_{\frac{5}{6}}\left(\frac{\xi H_o t}{\sqrt{2}}\right)
$$

$$
\Rightarrow \sim a + \text{oscillations}, \tag{B.11}
$$

where ${}_2F_1(a, b; c, z)$, is the Gaussian hypergeometric function and $J_n(z)$ is the Bessel function of the first kind of order n. In the second line of (B.11) we have included only the approximate solution for the growing mode, while in the third line we have taken into account just the dominating small-argument limit of the Bessel function. As expected, we thus find again that to leading order $d \log \hat{\delta}/d \log a = 1$, but there can occur slight modifications as $\xi H_0 t$ begins to grow and become non-negligible. The details of the corresponding possible signatures in the matter power spectrum, and their impact to the constraints on the specific model, need to be computed by integrating the equations numerically.

Finally, we will look at the same matter-scaling solution as for the previous model. At the fixed point B of Ref. [44], Eq. (4.27) becomes

$$
\ddot{\hat{\delta}} + \left[\frac{4}{3t} - \frac{6\lambda\xi H_0}{3\lambda^4\xi^2 H_0^2 t^2 - 6\lambda^3\xi H_0 t + 4(\lambda^2 - 3)^2}\right]\dot{\hat{\delta}} = \frac{8(\lambda^2 - 3)^4}{3\lambda^4\left[3\lambda^4\xi^2 H_0^2 t^2 - 6\lambda^3\xi H_0 t + 4(\lambda^2 - 3)^2\right]t^2}\hat{\delta},
$$

(B.12)

where, as explained in subsection 4.2.1, due to the choices $w_0 = c_s^2 = 0$, $f \sim 1/n$, there is no scale-dependence. For completeness and transparency, we solve this equation numerically, and in Fig. 2 we present the resulting growth-rate evolution as a function of the scale factor, for six combinations of the model parameters λ and ξ. As we observe, at high redshifts the
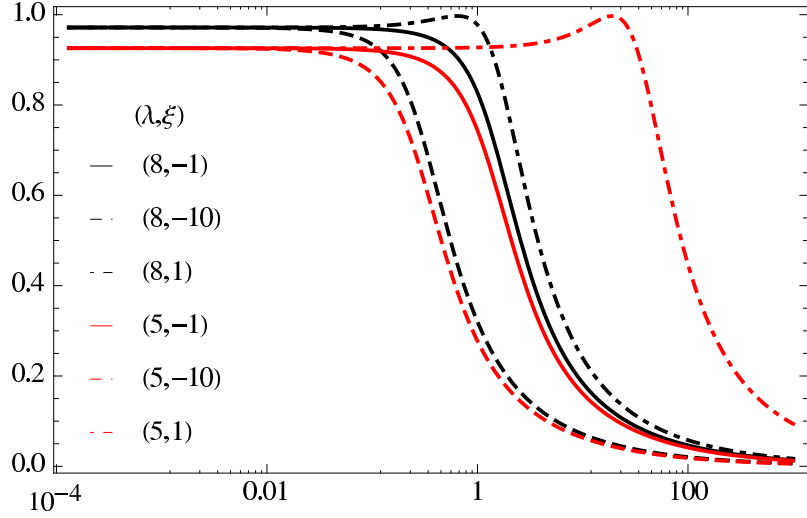
Figure 2. *The growth rate for matter densities for the model (B.9) at the matter-scaling background solution (fixed point B of [44]), as a function of the scale factor, for six combinations of the model parameters λ and ξ as indicated in the legend. The solutions begin from the constant growth rate given as the dominating solution to (B.7).*

solutions begin from the constant value given as the growing solution to Eq. (B.7). As the coupling becomes dynamically significant, the growth rate begins to evolve non-trivially, until the perturbations eventually freeze. However, the late evolution of the solutions should not be trusted as precisely when the coupling becomes significant, since the background solutions begin to deviate from the matter-scaling fixed point solution that was assumed in (B.12). Results from full evolution of the perturbations and systematic investigation of the parameter space of viable models, will hopefully be presented elsewhere.

C   Generalizations of the derivative couplings

In this appendix we briefly consider the applicability of the scalar-fluid formalism within the context of so called disformal couplings [41, 42, 47, 59, 73]. In particular, we propose some possible generalizations of the scalar-fluid framework, in order to incorporate more general derivative-type interactions terms than the one in (2.21) in which we restricted to in this work. The disformal relation is a transformation of the metric of the form

$$\tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu} + D(\phi)\phi_{,\mu}\phi_{,\nu} \,. \tag{C.1}$$

The functions C and D could depend also upon the kinetic term of the scalar field, $X = g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$, but for the purposes of the present discussion our simple points can be deduced as well from the less general starting point (C.1).

Let us first consider just the scalar-fluid - formalism coupled in the disformal metric.

Then Eqs. (2.20),(2.21) generalize to

$$
\begin{aligned}
\tilde{\mathcal{L}}_M &= -\sqrt{-\tilde{g}}\,\rho(\tilde{n}) + f(\tilde{n},\phi)\tilde{g}_{\mu\nu}J^\mu\phi^{,\mu} + J \\
&= \sqrt{-g}C^2(\phi)\gamma^{-1}(\phi,X)\rho(\tilde{n},s) + C(\phi)\gamma^{-2}(\phi,X)f(\tilde{n},\phi)g_{\mu\nu}J^\mu\phi^{,\mu} + J\,. \quad \text{(C.2)}
\end{aligned}
$$

Here we have defined the function $\gamma(\phi,X)$ as $\gamma = \left(1 + \frac{D}{C}X\right)^{-\frac{1}{2}}$. This already suggests that the functions ρ and f in Eqs. (2.20),(2.21) could be considered to be functions of X as well[11]. The number density in the disformal frame is given by

$$
\tilde{n} = C^{-\frac{3}{2}}(\phi)\gamma(\phi,X)\sqrt{1 + \frac{C(\phi)\,(\phi_{,\mu}J^\mu)^2}{D(\phi)g_{\mu\nu}J^\mu J^\nu}}\,n \quad \text{(C.3)}
$$

In practice it would be definitely easier to work in the disformal frame and obtain precisely the same sets of equations as in subsection 2.1, but with the appropriate tilde quantities, and then restore the metric $g_{\mu\nu}$ using (C.1).

We can also follow the opposite approach, by starting from a disformal field theory and consider it in terms of the scalar-fluid formalism. As an example, we consider the Lagrangian of a test particle living in a disformal metric:

$$
\tilde{\mathcal{L}} = m_0\sqrt{-\tilde{g}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\delta^{(4)}\left(x - x(\lambda)\right)\,. \quad \text{(C.4)}
$$

Varying the Lagrangian with respect to the metric $g_{\mu\nu}$ we obtain the stress energy tensor (2.24) of a pressureless perfect fluid, $T_{\mu\nu} = \rho U_\mu U_\nu$, where we have made the identifications

$$
\begin{aligned}
\rho &= m(\phi)\sqrt{\frac{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{g}}\left[1 + \frac{D(\phi)}{C(\phi)}\left(\phi_{,\alpha}U^\alpha\right)^2\right]^{-\frac{1}{2}}\delta^{(4)}\left(x - x(\lambda)\right), \\
U^\mu &= \frac{\dot{x}^\mu}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\,. \quad \text{(C.5)}
\end{aligned}
$$

We have denoted the time-dependent effective mass by $m(\phi) = \sqrt{C(\phi)}m_0$, and since the stress energy density is that of a dust-like component, we are allowed to make the straightforward interpretation that $\rho = m(\phi)n$. We can then - at least formally - write

$$
\tilde{\mathcal{L}} = \sqrt{-g}\rho(\phi,n)\left[1 + \frac{D(\phi)}{C(\phi)}\left(\phi_{,\alpha}U^\alpha\right)^2\right] \quad \Rightarrow \quad \sqrt{-g}m(\phi)n + \frac{D(\phi)}{C(\phi)}\frac{m(\phi)}{n}\frac{(J^\mu\phi_{,\mu})^2}{\sqrt{-g}}\,. \quad \text{(C.6)}
$$

The second form follows by naively using the relations of the fluid variables deduced in Section 2. Though this may not result in a completely equivalent theory in the scalar-fluid formalism, it clearly suggests the extension of the latter to allow for nonlinear derivative couplings between the fluid four-velocity and the gradient of the scalar field. One notes that the novel term appears naturally with the desired inverse proportionality to the number density (the metric determinant comes only to adjust the weight correctly).

---

[11]Note that the constraints are not affected since the metric decouples completely from J. However, one could formulate the theory in terms of J_μ instead, or consider $j^\mu = \sqrt{-g}J^\mu$, and thus the disformal relation (C.1) would result in quite non-trivial modifications.

References

[1] S. Nojiri and S. D. Odintsov, *Introduction to modified gravity and gravitational alternative for dark energy*, eConf **C0602061**, 06 (2006), Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007), [arXiv:hep-th/0601213].

[2] S. Capozziello and M. De Laurentis, *Extended Theories of Gravity*, Phys. Rept. **509**, 167 (2011) [arXiv:1108.6266].

[3] E. J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, Int. J. Mod. Phys. D **15**, 1753 (2006), [arXiv:hep-th/0603057].

[4] Y. F. Cai, E. N. Saridakis, M. R. Setare and J. Q. Xia, *Quintom Cosmology: Theoretical implications and observations*, Phys. Rept. **493**, 1 (2010), [arXiv:0909.2776].

[5] B. Ratra and P. J. E. Peebles, *Cosmological Consequences of a Rolling Homogeneous Scalar Field*, Phys. Rev. D **37**, 3406 (1988).

[6] C. Wetterich, *Cosmology and the Fate of Dilatation Symmetry*, Nucl. Phys. B **302**, 668 (1988).

[7] A. R. Liddle and R. J. Scherrer, *A Classification of scalar field potentials with cosmological scaling solutions*, Phys. Rev. D **59**, 023509 (1999), [arXiv:astro-ph/9809272].

[8] N. Tamanini, *Dynamics of cosmological scalar fields*, Phys. Rev. D **89**, no. 8, 083521 (2014) [arXiv:1401.6339].

[9] C. Q. Geng, M. W. Hossain, R. Myrzakulov, M. Sami and E. N. Saridakis, *Quintessential inflation with canonical and noncanonical scalar fields and Planck 2015 results*, [arXiv:1502.03597].

[10] S. Nojiri, S. D. Odintsov, V. K. Oikonomou and E. N. Saridakis, *Singular cosmological evolution using canonical and phantom scalar fields*, [arXiv:1503.08443].

[11] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, *Essentials of k essence*, Phys. Rev. D **63**, 103510 (2001), [arXiv:astro-ph/0006373].

[12] J. -P. Uzan, *Cosmological scaling solutions of nonminimally coupled scalar fields*, Phys. Rev. D **59**, 123510 (1999), [arXiv:gr-qc/9903004].

[13] R. de Ritis, A. A. Marino, C. Rubano and P. Scudellaro, *Tracker fields from nonminimally coupled theory*, Phys. Rev. D **62**, 043506 (2000), [arXiv:hep-th/9907198].

[14] V. Faraoni, *Inflation and quintessence with nonminimal coupling*, Phys. Rev. D **62**, 023504 (2000), [arXiv:gr-qc/0002091].

[15] E. Elizalde, S. Nojiri and S. D. Odintsov, *Late-time cosmology in (phantom) scalar-tensor theory: Dark energy and the cosmic speed-up*, Phys. Rev. D **70**, 043539 (2004) [arXiv:hep-th/0405034].

[16] S. Nojiri, S. D. Odintsov and P. V. Tretyakov, *Dark energy from modified F(R)-scalar-Gauss-Bonnet gravity*, Phys. Lett. B **651**, 224 (2007) [arXiv:0704.2520].

[17] T. Koivisto and D. F. Mota, *Cosmology and Astrophysical Constraints of Gauss-Bonnet Dark Energy*, Phys. Lett. B **644**, 104 (2007) [arXiv:astro-ph/0606078].

[18] T. Koivisto and D. F. Mota, *Gauss-Bonnet Quintessence: Background Evolution, Large Scale Structure and Cosmological Constraints*, Phys. Rev. D **75**, 023518 (2007) [arXiv:hep-th/0609155].

[19] K. Enqvist, T. Koivisto and G. Rigopoulos, JCAP **1205**, 023 (2012) [arXiv:1107.3739 [astro-ph.CO]].

[20] C. -Q. Geng, C. -C. Lee, E. N. Saridakis and Y. -P. Wu, *'Teleparallel' Dark Energy*, Phys. Lett. B **704**, 384 (2011), [`arXiv:1109.1092`].

[21] C. -Q. Geng, C. -C. Lee, E. N. Saridakis, *Observational Constraints on Teleparallel Dark Energy*, JCAP **1201**, 002 (2012), [`arXiv:1110.0913`].

[22] L. Amendola, *Cosmology with nonminimal derivative couplings*, Phys. Lett. B **301**, 175 (1993), [`arXiv:gr-qc/9302010`].

[23] E. N. Saridakis and S. V. Sushkov, *Quintessence and phantom cosmology with non-minimal derivative coupling*, Phys. Rev. D **81**, 083510 (2010), [`arXiv:1002.3478`].

[24] T. S. Koivisto and N. J. Nunes, *Coupled three-form dark energy*, Phys. Rev. D **88**, no. 12, 123512 (2013) [`arXiv:1212.2541`].

[25] G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, Int. J. Theor. Phys. 10, 363-384 (1974).

[26] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, *From k-essence to generalised Galileons*, Phys. Rev. D **84**, 064039 (2011), [`arXiv:1103.3260`].

[27] J. Gleyzes, D. Langlois and F. Vernizzi, *A unifying description of dark energy*, Int. J. Mod. Phys. D **23**, no. 13, 1443010 (2015) [`arXiv:1411.3712`].

[28] D. Bettoni and M. Zumalacrregui, *Kinetic mixing in scalar-tensor theories of gravity*, Phys. Rev. D **91**, no. 10, 104009 (2015) [`arXiv:1502.02666`].

[29] C. Wetterich, *The Cosmon model for an asymptotically vanishing time dependent cosmological 'constant'*, Astron. Astrophys. **301**, 321 (1995) [`arXiv:hep-th/9408025`].

[30] L. Amendola, *Scaling solutions in general nonminimal coupling theories*, Phys. Rev. D **60**, 043501 (1999) [`arXiv:astro-ph/9904120`].

[31] G.R. Farrar and P.J.E. Peebles, *Interacting dark matter and dark energy*, Astrophys. J. **604**, 1 (2004) [`arXiv:astro-ph/0307316`].

[32] T. Koivisto, *Growth of perturbations in dark matter coupled with quintessence*, Phys. Rev. D **72**, 043516 (2005) [`arXiv:astro-ph/0504571`].

[33] O. E. Bjaelde, A. W. Brookfield, C. van de Bruck, S. Hannestad, D. F. Mota, L. Schrempp and D. Tocchini-Valentini, *Neutrino Dark Energy – Revisiting the Stability Issue*, JCAP **0801**, 026 (2008) [`arXiv:0705.2018`].

[34] C. G. Boehmer, G. Caldera-Cabral, R. Lazkoz and R. Maartens, *Dynamics of dark energy with a coupling to dark matter*, Phys. Rev. D **78**, 023505 (2008) [`arXiv:0801.1565`].

[35] M. Jamil, E. N. Saridakis and M. R. Setare, *Thermodynamics of dark energy interacting with dark matter and radiation*, Phys. Rev. D **81**, 023007 (2010) [`arXiv:0910.0822`].

[36] B. Li and J. D. Barrow, *N-Body Simulations for Coupled Scalar Field Cosmology*, Phys. Rev. D **83**, 024007 (2011) [`arXiv:1005.4231`].

[37] X. m. Chen, Y. Gong, E. N. Saridakis, *Time-dependent interacting dark energy and transient acceleration*, Int. J. Theor. Phys. **53**, 469 (2014) [`arXiv:1111.6743`].

[38] V. Pettorino, L. Amendola, C. Baccigalupi and C. Quercellini, *Constraints on coupled dark energy using CMB data from WMAP and SPT*, Phys. Rev. D **86**, 103507 (2012)

[arXiv:1207.3293].

[39] Y. L. Bolotin, A. Kostenko, O. A. Lemets and D. A. Yerokhin, *Cosmological Evolution With Interaction Between Dark Energy And Dark Matter*, Int. J. Mod. Phys. D **24**, no. 03, 1530007 (2014) [arXiv:1310.0085].

[40] C. Llinares and D. F. Mota, *Cosmological simulations of screened modified gravity out of the static approximation: effects on matter distribution*, Phys. Rev. D **89**, no. 8, 084023 (2014) [arXiv:1312.6016].

[41] C. van de Bruck and J. Morrice, *Disformal couplings and the dark sector of the universe*, JCAP **1504**, no. 04, 036 (2015) [arXiv:1501.03073].

[42] R. Hagala, C. Llinares and D. F. Mota, *Cosmological simulations with disformally coupled symmetron fields*, [arXiv:1504.07142].

[43] C. G. Boehmer, N. Tamanini and M. Wright, *Interacting quintessence from a variational approach Part I: algebraic couplings*, [arXiv:1501.06540].

[44] C. G. Boehmer, N. Tamanini and M. Wright, *Interacting quintessence from a variational approach Part II: derivative couplings*, [arXiv:1502.04030].

[45] J. D. Brown, *Action functionals for relativistic perfect fluids*, Class. Quant. Grav. **10**, 1579 (1993) [arXiv:gr-qc/9304026].

[46] D. Bettoni, S. Liberati and L. Sindoni, *Extended LCDM: generalized non-minimal coupling for dark matter fluids*, JCAP **1111**, 007 (2011) [arXiv:1108.1728].

[47] D. Bettoni and S. Liberati, *Dynamics of non-minimally coupled perfect fluids*, [arXiv:1502.06613].

[48] V. Faraoni, J. B. Dent and E. N. Saridakis, *Covariantizing the interaction between dark energy and dark matter*, Phys. Rev. D **90**, no. 6, 063510 (2014) [arXiv:1405.7288].

[49] G. Olivares, F. Atrio-Barandela and D. Pavon, *Observational constraints on interacting quintessence models*, Phys. Rev. D **71**, 063523 (2005) [arXiv:astro-ph/0503242].

[50] Z. K. Guo, N. Ohta and S. Tsujikawa, *Probing the Coupling between Dark Components of the Universe*, Phys. Rev. D **76**, 023508 (2007) [arXiv:astro-ph/0702015].

[51] J. Valiviita, E. Majerotto and R. Maartens, *Instability in interacting dark energy and dark matter fluids*, JCAP **0807**, 020 (2008) [arXiv:0804.0232].

[52] B. M. Jackson, A. Taylor and A. Berera, Phys. Rev. D **79**, 043526 (2009) [arXiv:0901.3272 [astro-ph.CO]].

[53] M. B. Gavela, D. Hernandez, L. Lopez Honorez, O. Mena and S. Rigolin, *Dark coupling*, JCAP **0907**, 034 (2009) [JCAP **1005**, E01 (2010)] [arXiv:0901.1611].

[54] J. Valiviita, R. Maartens and E. Majerotto, Mon. Not. Roy. Astron. Soc. **402**, 2355 (2010) [arXiv:0907.4987 [astro-ph.CO]].

[55] D. G. A. Duniya, D. Bertacca and R. Maartens, Phys. Rev. D **91**, 063530 (2015) [arXiv:1502.06424 [astro-ph.CO]].

[56] J. Valiviita and E. Palmgren, *Distinguishing interacting dark energy from wCDM with CMB, lensing, and baryon acoustic oscillation data*, [arXiv:1504.02464].

[57] X. m. Chen, Y. g. Gong and E. N. Saridakis, *Phase-space analysis of interacting phantom cosmology*, JCAP **0904**, 001 (2009) [arXiv:0812.1117].

[58] N. Tamanini, *On phenomenological models of dark energy interacting with dark matter*, [arXiv:1504.07397].

[59] T. Koivisto, D. Wills and I. Zavala, *Dark D-brane Cosmology*, JCAP **1406**, 036 (2014) [arXiv:1312.2597].

[60] J. Gleyzes, D. Langlois, M. Mancarella and F. Vernizzi, *Effective Theory of Interacting Dark Energy*, [arXiv:1504.05481].

[61] A. Pourtsidou, C. Skordis and E. J. Copeland, *Models of dark matter coupled to dark energy*, Phys. Rev. D **88**, no. 8, 083505 (2013) [arXiv:1307.0458].

[62] C. Skordis, A. Pourtsidou and E. J. Copeland, *Parametrized post-Friedmannian framework for interacting dark energy theories*, Phys. Rev. D **91**, no. 8, 083537 (2015) [arXiv:1502.07297].

[63] J. R. Ray, *Lagrangian Density for Perfect Fluids in General Relativity*, J. Math. Phys. **13**, 1451 (1972).

[64] N. Andersson and G. L. Comer, *Relativistic fluid dynamics: Physics for many different scales*, Living Rev. Rel. **10**, 1 (2007) [arXiv:gr-qc/0605010].

[65] G. Ballesteros, B. Bellazzini and L. Mercolli, *The effective field theory of multi-component fluids*, JCAP **1405**, 007 (2014) [arXiv:1312.2957].

[66] G. Ballesteros, *The effective theory of fluids at NLO and implications for dark energy*, JCAP **1503**, no. 03, 001 (2015) [arXiv:1410.2793].

[67] A. Y. Kamenshchik, U. Moschella and V. Pasquier, *An Alternative to quintessence*, Phys. Lett. B **511**, 265 (2001) [arXiv:gr-qc/0103004].

[68] M. C. Bento, O. Bertolami and A. A. Sen, *Generalized Chaplygin gas, accelerated expansion and dark energy matter unification*, Phys. Rev. D **66**, 043507 (2002) [arXiv:gr-qc/0202064].

[69] K. Freese and M. Lewis, *Cardassian expansion: A Model in which the universe is flat, matter dominated, and accelerating*, Phys. Lett. B **540**, 1 (2002) [arXiv:astro-ph/0201229].

[70] T. Koivisto, *Covariant conservation of energy momentum in modified gravities*, Class. Quant. Grav. **23**, 4289 (2006) [arXiv:gr-qc/0505128].

[71] N. Tamanini and T. S. Koivisto, *Consistency of nonminimally coupled $f(R)$ gravity*, Phys. Rev. D **88**, no. 6, 064052 (2013) [arXiv:1308.3401].

[72] T. Harko and F. S. N. Lobo, *Generalized curvature-matter couplings in modified gravity*, Galaxies **2**, no. 3, 410 (2014) [arXiv:1407.2013].

[73] T. S. Koivisto, D. F. Mota and M. Zumalacarregui, *Screening Modifications of Gravity through Disformally Coupled Fields*, Phys. Rev. Lett. **109**, 241102 (2012) [arXiv:1205.3167].

[74] L. Amendola, S. Tsujikawa and M. Sami, *Phantom damping of matter perturbations*, Phys. Lett. B **632**, 155 (2006) [arXiv:astro-ph/0506222].

[75] H. Sandvik, M. Tegmark, M. Zaldarriaga and I. Waga, *The end of unified dark matter?*, Phys. Rev. D **69**, 123524 (2004) [arXiv:astro-ph/0212114].

[76] T. Koivisto, H. Kurki-Suonio and F. Ravndal, *The CMB spectrum in Cardassian models*, Phys. Rev. D **71**, 064027 (2005) [arXiv:astro-ph/0409163].

[77] T. Koivisto and H. Kurki-Suonio, *Cosmological perturbations in the palatini formulation of modified gravity*, Class. Quant. Grav. **23**, 2355 (2006) [arXiv:astro-ph/0509422].

[78] T. Koivisto, *The matter power spectrum in f(r) gravity*, Phys. Rev. D **73**, 083517 (2006) [arXiv:astro-ph/0602031].

[79] T. Koivisto, *Viable Palatini-f(R) cosmologies with generalized dark matter*, Phys. Rev. D **76**, 043527 (2007) [arXiv:0706.0974].